

Notes on a model theory of quantum 2-torus T_q^2 for generic q

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Abstract

We describe a structure over the complex numbers associated with the non-commutative algebra \mathcal{A}_q called quantum 2-tori. These turn out to have uncountably categorical $L_{\omega_1, \omega}$ -theory, and are similar to other pseudo-analytic structures considered by the second author. The first-order theory of a quantum torus for generic q interprets arithmetic and so is unstable and undecidable. But certain interesting reduct of the structure, a quantum line bundle, is superstable.

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1 Introduction

Quantum tori have been considered in various contexts;

1. Generalizing the definition of algebraic tori one obtains the notion of *quantum n -torus* over an abstract field \mathbb{F} as the \mathbb{F} -algebra $\mathcal{O}_q((\mathbb{F}^\times)^n)$ with generators x_1^\pm, \dots, x_n^\pm and the relation

$$x_i x_j = q x_j x_i.$$

See [2].

2. More often one is interested in a generalization of tori in the context of real structure on complex manifolds. Then the appropriate generalization to non-commutative setting is based on the notion of a C^* -algebra. Namely, in the above example we

would assume that $\mathbb{F} = \mathbb{C}$ and $\mathcal{O}_q((\mathbb{F}^\times)^n)$ is a C^* -algebra (that is a normed algebra with an involution $x \mapsto x^*$, where x^* is read as an operator adjoint to x), and the x_i are unitary, that is $x^* = x^{-1}$. See [3]

3. Many more beautiful and important examples can be seen as generalizations of the above, including quantum groups.

Classically, in non-commutative geometry one studies representation theory of the algebras in question. In case 1, finite-dimensional representations in \mathbb{F} -vector spaces, and in case 2, representations in Hilbert spaces. One of the main suppositions of non-commutative geometry is that unlike commutative case there is no geometric object corresponding to the quantum algebra, and the best we can have in place of the Gel'fand-Naimark duality (co-ordinate algebra – geometric space) is the correspondence between the algebra and its category of representations.

In this paper, however, we construct geometric objects, which can be seen as representing the information coded in the algebra of a quantum 2-torus. This is similar to what has been done by the second author in [11] in the case when q is a root of unity. In that case the appropriate geometric object is a Noetherian Zariski geometry, and when the algebra in question is commutative the Zariski geometry is just an affine algebraic variety, as in the classical duality.

The object $T_q^2(\mathbb{F})$ constructed in this paper over an algebraically closed field \mathbb{F} hopefully can be classified as an analytic Zariski geometry in the sense of [10], but we don't prove this fact here. Our main result, apart from construction as such, is that a simple $L_{\omega_1, \omega}$ -sentence characterizing $T_q^2(\mathbb{F})$ is categorical in uncountable cardinals.

Let Γ denote an infinite cyclic group generated by an element $q \in \mathbb{F}^\times$. We denote \mathcal{A}_q the non-commutative algebra $\mathcal{O}_q((\mathbb{F}^\times)^2)$ with generators written as U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV.$$

Our objective is to construct a structure $T_q^2(\mathbb{F})$ which interprets U, U^{-1}, V, V^{-1} as operators acting on it and thus represents the algebra \mathcal{A}_q .

In section 2, we first construct non-commutative geometric objects called Γ -bundles and line-bundles. Then we construct the *quantum* 2-torus $T_q^2(\mathbb{F})$ associated with the algebra \mathcal{A}_q over the algebraically closed field \mathbb{F} having two quantum line-bundles with a pairing function.

After constructing $T_q^2(\mathbb{F})$, in section 3 we study its properties from model theoretic point of view.

Three main theorems proved in this paper are;

1. $T_q^2(\mathbb{F})$ is axiomatisable by an $L_{\omega_1, \omega}$ -sentence which is categorical in uncountable cardinals. (Theorem 8)
2. The first-order theory of line-bundles, which is a reduct of $T_q^2(\mathbb{F})$, is superstable. (Theorem 15)
3. The first-order theory of $T_q^2(\mathbb{F})$, for q not a root of unity, interprets the ring of integers. Hence the theory of the quantum 2-torus is undecidable and unstable. (Theorem 17)

Prerequisites in model theory is minimal and all found in standard text books such as [5] or [7]. Let $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N}$ denote the set of complex numbers, the set of real numbers, the set of integers, and the set of natural numbers, respectively. All the fields considered in this paper are of characteristic zero.

1.1 Further works

In a forthcoming work we would like to classify $T_q^2(\mathbb{F})$ as an analytic Zariski structure, just like [11] classifies a quantum torus at root of unity as Noetherian Zariski structure. This requires a quantifier elimination statement and a detailed analysis of definability, this time including $\mathcal{L}_{\omega_1, \omega}$ -formulae.

A very important model theoretic next step in the study of quantum tori would be the study of definable bijections between $T_q^2(\mathbb{F})$ and $T_{q'}^2(\mathbb{F})$, analogue of regular isomorphisms between algebraic varieties in algebraic geometry. Here "definable" assumes definability in an ambient larger structure over a field \mathbb{F} , where the $T_q^2(\mathbb{F})$ are definable for each choice of q . Such a definable bijection (isomorphisms of tori) at the level of coordinate algebra $\mathcal{O}_q((\mathbb{F}^\times)^n)$ must correspond to a Morita equivalence between algebras.

2 The quantum 2-torus $T_q^2(\mathbb{F})$

Let \mathbb{F} be a field, and $q \in \mathbb{F}^*$, not a root of unity. Consider a \mathbb{F} -algebra \mathcal{A}_q generated by operators U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV, \quad UU^{-1} = U^{-1}U = VV^{-1} = V^{-1}V = I.$$

Let $\Gamma_q = q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\}$ be a cyclic multiplicative subgroup of \mathbb{F}^* . From now on in this section we work in an uncountable \mathbb{F} -module \mathcal{M} such that $\dim \mathcal{M} \geq |\mathbb{F}|$. Also we drop the subscript q from Γ_q when it is clear from the context.

2.1 Γ -sets, Γ -bundles, line-bundles

For each pair $(u, v) \in \mathbb{F}^* \times \mathbb{F}^*$, we will construct two \mathcal{A}_q -modules $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ so that both $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ are sub-modules of \mathcal{M} .

Before starting the construction of $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ for general q , it is important to keep in mind the case when q is actually a root of unity. In such case we can also define $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ in the same manner described in this paper, however $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ will be inter-definable uniformly on the pair (u, v) .

We now start the construction with q not a root of unity. The module $M_{|u,v\rangle}$ is generated by linearly independent elements labelled $\{\mathbf{u}(\gamma u, v) \in \mathcal{M} : \gamma \in \Gamma\}$ satisfying

$$\begin{aligned} U & : \mathbf{u}(\gamma u, v) \mapsto \gamma u \mathbf{u}(\gamma u, v), \\ V & : \mathbf{u}(\gamma u, v) \mapsto v \mathbf{u}(q^{-1} \gamma u, v), \end{aligned} \tag{1}$$

and also

$$\begin{aligned} U^{-1} & : \mathbf{u}(\gamma u, v) \mapsto \gamma^{-1} u^{-1} \mathbf{u}(\gamma u, v), \\ V^{-1} & : \mathbf{u}(\gamma u, v) \mapsto v^{-1} \mathbf{u}(q \gamma u, v). \end{aligned} \tag{2}$$

Next let $\phi : \mathbb{F}^*/\Gamma \rightarrow \mathbb{F}^*$ such that $\phi(x\Gamma) \in x\Gamma$ for each $x\Gamma \in \mathbb{F}^*/\Gamma$. Put $\Phi = \text{ran}(\phi)$. We call ϕ a *choice function* and Φ the system of representatives.

Set for $\langle u, v \rangle \in \Phi^2$

$$\begin{aligned} \Gamma \cdot \mathbf{u}(u, v) & := \{\gamma \mathbf{u}(u, v) : \gamma \in \Gamma\}, \\ \mathbf{U}_{\langle u, v \rangle} & := \bigcup_{\gamma \in \Gamma} \Gamma \cdot \mathbf{u}(\gamma u, v) = \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \gamma_1, \gamma_2 \in \Gamma\}. \end{aligned} \tag{3}$$

And set

$$\begin{aligned} \mathbf{U}_\phi & := \bigcup_{\langle u, v \rangle \in \Phi^2} \mathbf{U}_{\langle u, v \rangle} \\ & = \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u, v \rangle \in \Phi^2, \gamma_1, \gamma_2 \in \Gamma\}, \\ \mathbb{F}^* \mathbf{U}_\phi & := \{x \cdot \mathbf{u}(\gamma u, v) : \langle u, v \rangle \in \Phi^2, x \in \mathbb{F}^*, \gamma \in \Gamma\}. \end{aligned} \tag{4}$$

Note: The notation like $x \cdot \mathbf{u}(\gamma u, v)$ above should be read as a 4-tuple (x, γ, u, v) .

We call $\Gamma \cdot \mathbf{u}(u, v)$ a Γ -**set** over the pair (u, v) , \mathbf{U}_ϕ a Γ -bundle over $\mathbb{F}^* \times \mathbb{F}^* / \Gamma$, and $\mathbb{F}^* \mathbf{U}_\phi$ a **line-bundle** over \mathbb{F}^* . Notice that \mathbf{U}_ϕ can also be seen as a bundle inside $\bigcup_{\langle u, v \rangle} M_{|u, v\rangle}$. Furthermore the line bundle $\mathbb{F}^* \mathbf{U}_\phi$ is closed under the action of the operators U and V satisfying the relations (1) and (2).

We define the module $M_{\langle v, u \rangle}$ generated by linearly independent elements labelled $\{\mathbf{v}(\gamma v, u) \in \mathcal{M} : \gamma \in \Gamma\}$ satisfying

$$\begin{aligned} U &: \mathbf{v}(\gamma v, u) \mapsto u \mathbf{v}(q \gamma v, u), \\ V &: \mathbf{v}(\gamma v, u) \mapsto \gamma v \mathbf{v}(\gamma v, u), \end{aligned} \quad (5)$$

and

$$\begin{aligned} U^{-1} &: \mathbf{v}(\gamma v, u) \mapsto u^{-1} \mathbf{v}(q^{-1} \gamma v, u), \\ V^{-1} &: \mathbf{v}(\gamma v, u) \mapsto \gamma^{-1} v^{-1} \mathbf{v}(\gamma v, u). \end{aligned} \quad (6)$$

Similarly a Γ -set $\Gamma \cdot \mathbf{v}(v, u)$ over the pair (v, u) , a Γ -bundle \mathbf{V}_ϕ over $\mathbb{F}^* / \Gamma \times \mathbb{F}^*$, and $\mathbb{F}^* \mathbf{V}_\phi$ a **line-bundle** over \mathbb{F}^* are defined.

As before the Γ -bundle \mathbf{V}_ϕ can also be seen as a bundle inside $\bigcup_{\langle v, u \rangle} M_{\langle v, u \rangle}$.

In the next section we treat $\mathbb{F}^* \mathbf{U}_\phi$ as an object definable in the structure $(\mathbf{U}_\phi, \mathbf{V}_\phi, \mathbb{F})$. For this we introduce an equivalence relation E identifying $\gamma \in \Gamma$ as an element of \mathbb{F}^* . Thus $\mathbb{F}^* \mathbf{U}_\phi \simeq (\mathbb{F}^* \times \mathbf{U}_\phi) / E$ where for $(x, \gamma_1 \cdot \mathbf{u}(\gamma_3 u, v)), (x', \gamma_2 \cdot \mathbf{u}(\gamma_3 u, v)) \in \mathbb{F}^* \times \mathbf{U}_\phi$ define

$$\begin{aligned} (x, \gamma_1 \cdot \mathbf{u}(\gamma_3 u, v)) \sim_E (x', \gamma_2 \cdot \mathbf{u}(\gamma_3 u, v)) &\iff \\ \exists \gamma \in \Gamma (x' = x \gamma^{-1} \wedge \gamma_2 = \gamma \gamma_1) \end{aligned}$$

We will then consider that the two operators U and V are acting on this definable classes $\mathbb{F}^* \mathbf{U}_\phi$ in the next section.

Similarly for $\mathbb{F}^* \mathbf{V}_\phi$ and the actions of U, V .

Remark 1 *Relations (1) mean that elements $\mathbf{u}(\gamma u, v)$ are eigenvectors and γu are eigenvalues of the operator U .*

Having defined the line bundles $\mathbb{F}^* \mathbf{U}_\phi$ and $\mathbb{F}^* \mathbf{V}_\phi$, we realize that any particular properties of the element q or the choice function ϕ are not used. This means the following:

Proposition 2 *Let \mathbb{F}, \mathbb{F}' be fields and $q \in \mathbb{F}, q' \in \mathbb{F}'$ such that there is an field isomorphism i from \mathbb{F} to \mathbb{F}' sending q to q' . Then i can be extended to an isomorphism from the Γ -bundle \mathbf{U}_ϕ to the Γ' -bundle $\mathbf{U}_{\phi'}$ and also from the line-bundle $\mathbb{F}^* \mathbf{U}_\phi$ to the line-bundle $(\mathbb{F}^*)' \mathbf{U}_{\phi'}$. The same is true for the line-bundles $\mathbb{F}^* \mathbf{V}_\phi$ and $(\mathbb{F}')^* \mathbf{V}_{\phi'}$.*

In particular the isomorphism type of Γ -bundles and line-bundles does not depend on the choice function.

Proof: Let i be an isomorphism from \mathbb{F} to \mathbb{F}' sending q to q' . Set $i(x \cdot \mathbf{u}(\gamma u, v)) = i(x) \cdot \mathbf{u}(i(\gamma u), i(v))$. Then this defines an isomorphism from $\mathbb{F}^* \mathbf{U}_\phi$ to $(\mathbb{F}')^* \mathbf{U}_{\phi'}$. ■

2.2 Pairing function and quantum 2-torus

It is clear from the construction that there is no interactions between Γ -bundles \mathbf{U}_ϕ and \mathbf{V}_ϕ . We now introduce the notion of *pairing function* $\langle \cdot | \cdot \rangle$ which plays the rôle of an *inner product* of two Γ -bundles \mathbf{U}_ϕ and \mathbf{V}_ϕ :

$$\langle \cdot | \cdot \rangle : (\mathbf{V}_\phi \times \mathbf{U}_\phi) \cup (\mathbf{U}_\phi \times \mathbf{V}_\phi) \rightarrow \Gamma. \quad (7)$$

We would like two operators U, V to behave like *unitary operators* on Hilbert space. This requirement forces us to postulate the following:

1. $\langle \mathbf{u}(u, v) | \mathbf{v}(v, u) \rangle = 1$,
2. for each $r, s \in \mathbb{Z}$, $\langle U^r V^s \mathbf{u}(u, v) | U^r V^s \mathbf{v}(v, u) \rangle = 1$,
3. for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$,

$$\langle \gamma_1 \mathbf{u}(\gamma_2 u, v) | \gamma_3 \mathbf{v}(\gamma_4 v, u) \rangle = \langle \gamma_3 \mathbf{v}(\gamma_4 v, u) | \gamma_1 \mathbf{u}(\gamma_2 u, v) \rangle^{-1},$$

4. $\langle \gamma_1 \mathbf{u}(\gamma_2 u, v) | \gamma_3 \mathbf{v}(\gamma_4 v, u) \rangle = \gamma_1^{-1} \gamma_3 \langle \mathbf{u}(\gamma_2 u, v) | \mathbf{v}(\gamma_4 v, u) \rangle$, and
5. for $v' \notin \Gamma \cdot v$ or $u' \notin \Gamma \cdot u$, $\langle q^s \mathbf{v}(v', u) | q^r \mathbf{u}(u', v) \rangle$ is not defined.

Proposition 3 *The pairing function (7) defined above satisfies the following: for any $m, k, r, s \in \mathbb{N}$ we have*

$$\langle q^s \mathbf{v}(q^m v, u) | q^r \mathbf{u}(q^k u, v) \rangle = q^{r-s-km} \quad (8)$$

and

$$\langle q^r \mathbf{u}(q^k u, v) | q^s \mathbf{v}(q^m v, u) \rangle = q^{km+s-r} = \langle q^s \mathbf{v}(q^m v, u) | q^r \mathbf{u}(q^k u, v) \rangle^{-1}. \quad (9)$$

Proof: We only prove (8). For this, it is enough to notice that for each $r, s \in \mathbb{Z}$, $\langle U^r V^s \mathbf{u}(u, v) | U^r V^s \mathbf{v}(v, u) \rangle = 1$ implies that

$$\langle \mathbf{u}(q^r u, v) | \mathbf{v}(q^s v, u) \rangle = q^{rs}. \quad \blacksquare$$

Finally we are ready to define the notion of quantum 2-torus:

Definition 4 We call the multi-sorted structure $(\mathbf{U}_\phi, \mathbf{V}_\phi, \langle \cdot | \cdot \rangle, \mathbb{F})$ with actions U, V with U satisfying (1), V satisfying (5) and $\langle \cdot | \cdot \rangle$ a pairing function defined as above a **quantum 2-torus** $T_q^2(\mathbb{F})$ over the field \mathbb{F} .

From Proposition 2 we know that the structure of the line-bundles does not depend on the choice function. The next proposition tells us that the structure of the quantum 2-torus $T_q^2(\mathbb{F})$ depends only on \mathbb{F} , q and not on the choice function.

Proposition 5 (cf. Proposition 4.4, [9]) Given $q \in \mathbb{F}^*$ not a root of unity, any two structures of the form $T_q^2(\mathbb{F})$ are isomorphic over \mathbb{F} . In other words, the isomorphism type of $T_q^2(\mathbb{F})$ does not depend on the system of representatives Φ .

Proof: Let ϕ, ψ be two choice functions of \mathbb{F}^*/Γ . Consider two structures $(\mathbf{U}, \mathbf{V})_\phi = (\mathbf{U}_\phi, \mathbf{V}_\phi)$ and $(\mathbf{U}, \mathbf{V})_\psi = (\mathbf{U}_\psi, \mathbf{V}_\psi)$. We show that these two structures are isomorphic over \mathbb{F} .

Suppose ϕ picks $\langle u_g, v_g \rangle$ from \mathbb{F}^*/Γ and ψ picks $\langle u_0, v_0 \rangle$ from the same coset of $\langle u_g, v_g \rangle$.

Consider the bases $\{\mathbf{u}(q^k u_g, v_g) : k \in \mathbb{Z}\}$ of $U_{\langle u_g, v_g \rangle}$ and $\{\mathbf{v}(q^k v_g, u_g) : k \in \mathbb{Z}\}$ of $V_{\langle u_g, v_g \rangle}$ in the structure $(\mathbf{U}_\phi, \mathbf{V}_\phi)$.

Since $\langle u_0, v_0 \rangle$ and $\langle u_g, v_g \rangle$ are in the same coset of \mathbb{F}^*/Γ there are $s, t \in \mathbb{Z}$ such that $u_0 = q^s u_g, v_0 = q^t v_g$.

We now want to transfer the structure of $\mathbf{U}_{\langle u_g, v_g \rangle}$ and $\mathbf{V}_{\langle u_g, v_g \rangle}$ to $\mathbf{U}_{\langle u_0, v_0 \rangle}$ and $\mathbf{V}_{\langle u_0, v_0 \rangle}$ respectively as follows. Set

- $\mathbf{u}(u_0, v_0) := q^{st} \mathbf{u}(q^s u_g, v_g),$
- $\mathbf{u}(q^k u_0, v_0) := v_0^k V^{-k} \mathbf{u}(u_0, v_0),$
- $\mathbf{v}(v_0, u_0) := \mathbf{v}(q^t v_g, u_g),$
- $\mathbf{v}(q^k v_0, u_0) := u_0^{-k} U^k \mathbf{v}(v_0, u_0),$

where $k \in \mathbb{Z}$. First notice that we have

$$\begin{aligned} \mathbf{u}(q^k u_0, v_0) &= v_0^k V^{-k} \mathbf{u}(u_0, v_0) \\ &= (q^t v_g)^k V^{-k} (q^{st} \mathbf{u}(q^s u_g, v_g)) \\ &= q^{kt} v_g^k q^{st} v_g^{-k} \mathbf{u}(q^{s+k} u_g, v_g) \\ &= q^{kt+st} \mathbf{u}(q^{k+s} u_g, v_g), \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}(q^k v_0, u_0) &= u_0^{-k} U^k \mathbf{v}(v_0, u_0) \\ &= q^{-ks} u_g^{-k} U^k \mathbf{v}(q^t v_g, u_g) \\ &= q^{-ks} u_g^{-k} u_g^k \mathbf{v}(q^{t+k} v_g, u_g) \\ &= q^{-sk} \mathbf{v}(q^{k+t} v_g, u_g). \end{aligned}$$

From these relations we see that the operators U and V act on the set $\{\mathbf{u}(q^k u_0, v_0) : k \in \mathbb{Z}\}$ according to the definition of \mathbf{U}_ϕ , that is

$$\begin{aligned}
U(\mathbf{u}(u_0, v_0)) &= U(q^{st} \mathbf{u}(q^s u_g, v_g)) \\
&= q^{st} U(\mathbf{u}(q^s u_g, v_g)) \\
&= q^{st} q^s u_g \mathbf{u}(q^s u_g, v_g) \\
&= u_0 q^{st} \mathbf{u}(q^s u_g, v_g) \\
&= u_0 \mathbf{u}(u_0, v_0), \\
V(\mathbf{u}(u_0, v_0)) &= V(q^{st} \mathbf{u}(q^s u_g, v_g)) \\
&= v_0 \mathbf{u}(q^{-1} u_0, v_0), \\
U(\mathbf{u}(q^k u_0, v_0)) &= U(v_0^k V^{-k} \mathbf{u}(u_0, v_0)) \\
&= q^k u_0 \mathbf{u}(q^k u_0, v_0), \\
V(\mathbf{u}(q^k u_0, v_0)) &= V(q^{kt+st} \mathbf{u}(q^{k+s} u_g, v_g)) \\
&= v_0 \mathbf{u}(q^{k-1} u_0, v_0).
\end{aligned}$$

By similar calculations we see that the operators U and V act on the set $\{\mathbf{v}(q^k v_0, u_0) : k \in \mathbb{Z}\}$ according to the definition of \mathbf{V}_ϕ .

Finally from the following relations we see that we can properly transfer the pairing function from $(\mathbf{U}_{\langle u_g, v_g \rangle}, \mathbf{V}_{\langle v_g, u_g \rangle})$ to $(\mathbf{U}_{\langle u_0, v_0 \rangle}, \mathbf{V}_{\langle v_0, u_0 \rangle})$:

$$\langle \mathbf{v}(v_0, u_0) | \mathbf{u}(u_0, v_0) \rangle = \langle \mathbf{v}(q^t v_g, u_g) | q^{st} \mathbf{u}(q^s u_g, v_g) \rangle = q^{st-st} = 1$$

and

$$\begin{aligned}
\langle \mathbf{v}(q^m v_0, u_0) | \mathbf{u}(q^k u_0, v_0) \rangle &= \langle q^{-sm} \mathbf{v}(q^{m+t} v_g, u_g) | q^{st+kt} \mathbf{u}(q^{k+s} u_g, v_g) \rangle \\
&= q^{st+kt-(-sm)-(m+t)(k+s)} \\
&= q^{-mk}.
\end{aligned}$$

We have now shown that the two structures $(\mathbf{U}_{\langle u_g, v_g \rangle}, \mathbf{V}_{\langle v_g, u_g \rangle})$ and $(\mathbf{U}_{\langle u_0, v_0 \rangle}, \mathbf{V}_{\langle v_0, u_0 \rangle})$ are isomorphic. Therefore so are the two structures $(\mathbf{U}, \mathbf{V})_\phi$ and $(\mathbf{U}, \mathbf{V})_\psi$. ■

From the above proposition we have as a corollary to Proposition 2 the following:

Corollary 6 *Let \mathbb{F}, \mathbb{F}' be fields and $q \in \mathbb{F}, q' \in \mathbb{F}'$ such that there is a field isomorphism i from \mathbb{F} to \mathbb{F}' sending q to q' . Then i can be extended to an isomorphism from the quantum 2-torus $T_q^2(\mathbb{F})$ to the quantum 2-torus $T_{q'}^2(\mathbb{F}')$.*

In particular the isomorphism type of quantum 2-torus does not depend on the choice function.

From now on we drop the subscript ϕ from line-bundles $\mathbf{U}_\phi, \mathbf{V}_\phi$ and write simply as $T_q^2(\mathbb{F}) = (\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F})$.

3 The model theory of quantum 2-torus over algebraically closed field

We now study the model theory of quantum 2-tori $T_q^2(\mathbb{F})$ under the assumption that \mathbb{F} is algebraically closed.

After introducing an appropriate language, we shall prove three theorems in this section;

1. $T_q^2(\mathbb{F})$ is axiomatisable by an $L_{\omega_1, \omega}$ -sentence Ψ which is categorical in uncountable cardinals. (Theorem 8)
2. The first-order theory of line-bundles, which is a reduct of $T_q^2(\mathbb{F})$, is superstable. (Theorem 15)
3. The first-order theory of $T_q^2(\mathbb{F})$, for q not a root of unity, interprets the ring of integers. Hence the theory of the quantum 2-torus is undecidable and unstable. (Theorem 17)

3.1 The language for quantum 2-tori

To define the sentence Ψ we introduce a language \mathcal{L}_q which is the language for multi-sorted structure $(\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F})$;

$$\mathcal{L}_q = \{+, \cdot, \mathbf{U}, \mathbf{V}, U, V, \langle \cdot | \cdot \rangle, \mathbb{F}, \Gamma, \pi\}$$

where $+, \cdot$ defined on \mathbb{F} and $\Gamma \subset \mathbb{F}$. Furthermore U, V are operators acting on \mathbf{U} and \mathbf{V} . Each $\gamma \in \Gamma$ acts on \mathbf{U} and \mathbf{V} . Also π is a function symbol which will be interpreted as a surjection from \mathbf{U} onto $\mathbb{F}^* \times \mathbb{F}^* / \Gamma$ and from \mathbf{V} onto $\mathbb{F}^* \times \mathbb{F}^* / \Gamma$.

3.2 $T_q^2(\mathbb{F})$ is $L_{\omega_1, \omega}$ -categorical in uncountable cardinals

Here we define the $\mathcal{L}_{\omega_1, \omega}$ -sentence Ψ in \mathcal{L}_q describing the quantum 2-torus $T_q^2(\mathbb{F}) = (\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F})$. Then we show that the sentence Ψ is categorical in uncountable cardinals.

Recall how we treat $\mathbb{F}^* \mathbf{U}$ as an object definable in the structure $(\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F})$. Similarly for $\mathbb{F}^* \mathbf{V}$. We extend that the two operators U and V are acting on these definable sets $\mathbb{F}^* \mathbf{U}$ and $\mathbb{F}^* \mathbf{V}$.

Let Ψ be the $L_{\omega_1, \omega}$ -sentence stating that

1. \mathbb{F} is an algebraically closed field of characteristic zero,

2. $q \in \mathbb{F}$ and not a root of unity,
3. Γ is a multiplicative subgroup of \mathbb{F} generated by q , i.e., $\Gamma \simeq q^{\mathbb{Z}}$,
4. π is surjective from \mathbf{U} onto $\mathbb{F}^* \times \mathbb{F}^*/\Gamma$,
5. for each $\gamma \in \Gamma$, $(u, v) \in \mathbb{F}^* \times \mathbb{F}^*/\Gamma$, $\pi^{-1}(\gamma u, v) \subset \mathbf{U}$ is generated by an element and
 - for each $\mathbf{u} \in \pi^{-1}(\gamma u, v)$ and $\gamma' \in \Gamma$, $\gamma' \mathbf{u} \in \pi^{-1}(\gamma u, v)$,
 - for each $\mathbf{u} \in \pi^{-1}(\gamma u, v)$ and $x \in \mathbb{F}^*$, $x\mathbf{u} \in \mathbb{F}^* \pi^{-1}(\gamma u, v)$,
6. $\mathbb{F}^* \mathbf{U}$, $\mathbb{F}^* \mathbf{V}$ are \mathbb{F} -modules,
7. operators U, V act on $\mathbb{F}^* \mathbf{U}$ and $\mathbb{F}^* \mathbf{V}$ according to (1) and (5), more precisely,
 - for each $\mathbf{u} \in \pi^{-1}(\gamma u, v)$, $x \in \mathbb{F}^*$ we have $U(x, \mathbf{u}) \in \mathbb{F}^* \mathbf{U}$, $\gamma u \mathbf{u} \in \pi^{-1}(\gamma u, v)$ and $U(x, \mathbf{u}) = \gamma u \mathbf{u}$,
 - for each $\mathbf{u} \in \pi^{-1}(\gamma u, v)$, $x \in \mathbb{F}^*$ we have $V(x, \mathbf{u}) \in \mathbb{F}^* \mathbf{V}$, there exists $\mathbf{u}' \in \pi^{-1}(q^{-1} \gamma u, v)$ and $V(x, \mathbf{u}) = x v \mathbf{u}'$,
8. the properties of the pairing function, more precisely,
 - for any $\mathbf{u} \in \pi^{-1}(u, v)$, $\mathbf{v} \in \pi^{-1}(v, u)$,
 - $\langle \mathbf{u}, \mathbf{v} \rangle = 1$,
 - for any $r, s \in \mathbb{Z}$, $\langle U^r V^s(\mathbf{u}), U^r V^s(\mathbf{v}) \rangle = 1$, where 1 is the multiplicative identity element of \mathbb{F} .
 - for any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$ and $\mathbf{u} \in \pi^{-1}(\gamma_2, v)$, $\mathbf{v} \in \pi^{-1}(\gamma_4 v, u)$,
 - $\langle \gamma_1 \mathbf{u}, \gamma_3 \mathbf{v} \rangle = \langle \gamma_3 \mathbf{v}, \gamma_1 \mathbf{u} \rangle^{-1}$ and,
 - $\langle \gamma_1 \mathbf{u}, \gamma_3 \mathbf{v} \rangle = \gamma_1^{-1} \gamma_3 \langle \mathbf{u}, \mathbf{v} \rangle$.

Lemma 7 *Let $M = (\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F})$ be an infinite \mathcal{L}_q -structure satisfying Ψ . Then M is a quantum 2-torus over \mathbb{F} .*

Proof: Let \mathbb{F} be an infinite algebraically closed field of characteristic zero, $q \in \mathbb{F}$ and $\Gamma = q^{\mathbb{Z}} \subset \mathbb{F}$ given by Ψ . Then \mathbf{U} and \mathbf{V} are the Γ -bundles defined over \mathbb{F} and q where operators U, V satisfy relations (1), (2), (5) and (6). We then construct line-bundles $\mathbb{F}^* \mathbf{U}$ and $\mathbb{F}^* \mathbf{V}$. Since the sentence Ψ describes the properties of the pairing function, from Proposition 2 we see that M is a quantum 2-torus over \mathbb{F} . ■

Thus Corollary 6 gives us

Theorem 8 *The $L_{\omega_1, \omega}$ -sentence Ψ is categorical in uncountable cardinals.*

Proof: Notice that any uncountable model of Ψ is a quantum 2-torus $T_q^2(\mathbb{F})$ over an uncountable algebraically closed field \mathbb{F} of characteristic zero. Since all such uncountable fields \mathbb{F} are isomorphic once we fix the cardinality, so are the quantum 2-tori $T_q^2(\mathbb{F})$ over such fields \mathbb{F} . Hence Ψ is categorical in uncountable cardinals. \blacksquare

3.3 Superstability of $\text{Th}(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$

From now on we study the first-order theoretic properties of quantum 2-tori. In this subsection we show that the first-order theory of $(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$ is axiomatizable and superstable. Key idea is that the predicate $\Gamma(x)$ describes the property of the set $q^{\mathbb{Z}}$ as a multiplicative subgroup with the following Lang-type property.

Definition 9 (Definition 2.3 [6]) *Let K be an algebraically closed field, and A a commutative algebraic group over K and Γ a subgroup of A . We say that (K, A, Γ) is of Lang-type if for every $n < \omega$ and every subvariety X (over K) of $A^n = A \times \cdots \times A$ (n times), $X \cap \Gamma^n$ is a finite union of cosets of subgroups of Γ^n .*

The Lang-type property gives us

Proposition 10 (Proposition 2.6 [6]) *Let K be an algebraically closed field, A a commutative algebraic group over K , and Γ a subgroup of A . Then (K, A, Γ) is of Lang-type if and only if $\text{Th}(K, +, \cdot, \Gamma, a)_{a \in K}$ is stable and $\Gamma(x)$ is one-based.*

Here $\Gamma(x)$ is *one based* means that for every n and every definable subset $X \subset \Gamma^n$, X is a finite boolean combination of cosets of definable subgroups of Γ^n .

With the above Definition 9 and Proposition 10 in mind, we axiomatize the properties of $(\mathbb{F}, +, \cdot, \Gamma)$ as follows;

Axioms for $(\mathbb{F}, +, \cdot, \Gamma)$

- A. 1 Γ satisfies the first order theory of a cyclic group with generator q ,
- A. 2 (Lang-type) for every n and every variety X of $(\mathbb{F}^*)^n$, $X \cap \Gamma^n$ is a finite union of cosets of definable subgroups of Γ^n .

Let $T_{\mathbb{F}, \Gamma}$ denote the set of all logical consequences of the axioms for Γ and ACF_0 axioms for the algebraically closed fields of characteristic zero.

Lemma 11 *The Lang-type property of $(\mathbb{F}, +, \cdot, \Gamma)$ is witnessed by its first-order theory.*

Proof: We may suppose X is irreducible. Each such variety $X \subset (\mathbb{F}^*)^n$ is definable by an irreducible polynomial $f(x_1, \dots, x_n)$ over \mathbb{F}^* . Definable cosets of Γ^n are of the form $\bar{\gamma}\Gamma^n = \gamma_1\Gamma \times \dots \times \gamma_n\Gamma$ where $\gamma_1, \dots, \gamma_n \in \Gamma(\mathbb{F})$. Hence the sentence " $X \cap \Gamma^n$ is a finite union of cosets of definable subgroups" is expressed as

$$(f(x_1, \dots, x_n) = 0) \wedge \Gamma(x_1) \wedge \dots \wedge \Gamma(x_n) \iff \bigvee_{i=1}^{N_f} \varphi_i(x_1, \dots, x_n).$$

Where each $\varphi_i(x_1, \dots, x_n)$ defines a coset. Crucial point here is that the number N_f of the bound of cosets is computable for each polynomial f . For this note first that for any $k \in \mathbb{N}$ the number of cosets of $q^{k\mathbb{Z}}$ in $q^{\mathbb{Z}}$ is k . Suppose

$$f(x_1, \dots, x_n) = \sum_{i=0}^{\deg(f)} \bar{a}_i \bar{x}_i^{m_i},$$

where each m_i is a multi index. Let M_i be the sum of multi index m_i . Then the bound N_f of number of cosets is $\deg(f) \cdot \sum_{i=0}^{\deg(f)} M_i$. Therefore the Lang-type property is first-order. \blacksquare

Proposition 12 $T_{\mathbb{F}, \Gamma}$ is complete. Hence $T_{\mathbb{F}, \Gamma} = \text{Th}(\mathbb{F}, +, \cdot, \Gamma)$.

Proof: Consider a saturated model $(\mathbb{F}, +, \cdot, \Gamma, q)$ of $T_{\mathbb{F}, \Gamma}$. Set $\Gamma(\mathbb{F}) = \{x \in \mathbb{F} : \mathbb{F} \models \Gamma(x)\}$. Let q be an element of \mathbb{F} interpreting the constant. By the axioms for Γ , $q^{\mathbb{Z}} \subset \Gamma(\mathbb{F}) \subset \mathbb{F}$.

Consider a complete type $t_0(x)$ generated by the following set of formulas,

$$t(x) = \{\Gamma(x), \exists y(x = qy), \exists y(x = q^2y), \dots\}.$$

By saturation there exists $\gamma_0 \in \Gamma(\mathbb{F})$ realizing $t_0(x)$. Clearly, $\gamma_0 \notin q^{\mathbb{Z}}$. Suppose elements $\gamma_0, \dots, \gamma_i \in \Gamma(\mathbb{F})$ have been defined. Let $t_{i+1}(x)$ be a complete type generated by the type $t(x)$ and the set

$$\{x \neq \gamma_0^{n_0}, \dots, x \neq \gamma_i^{n_i} : n_0, \dots, n_i \in \mathbb{Z}\}.$$

From saturation, we have $\gamma_{i+1} \in \Gamma(\mathbb{F})$ such that

$$\gamma_{i+1} \notin \bigcup_{l=0}^i \gamma_l^{\mathbb{Z}}.$$

In this way by saturation as before we see that there exist $\gamma_0, \gamma_1, \dots, \gamma_i, \dots \in \Gamma(\mathbb{F})$ ($i < |\mathbb{F}|$) such that

$$\Gamma(\mathbb{F}) = q^{\mathbb{Z}} \cup \bigcup_{i < |\mathbb{F}|} \gamma_i^{\mathbb{Z}}.$$

Now take two saturated models $(\mathbb{F}, +, \cdot, \Gamma, q)$ and $(\mathbb{F}', +, \cdot, \Gamma', q')$ of $T_{\mathbb{F}, \Gamma}$ of the same cardinality. There is an isomorphism i from \mathbb{F} to \mathbb{F}' sending q to q' . By the above formula for $\Gamma(\mathbb{F})$ and the back-and-forth argument we can extend i to have that $\Gamma(\mathbb{F}) \simeq \Gamma'(\mathbb{F}')$. Hence $(\mathbb{F}, +, \cdot, \Gamma, q)$ and $(\mathbb{F}', +, \cdot, \Gamma', q')$ are isomorphic as saturated models of $T_{\mathbb{F}, \Gamma}$. This completes the proof of the completeness of the theory $T_{\mathbb{F}, \Gamma}$. \blacksquare

Theorem 13 $T_{\mathbb{F}, \Gamma}$ is superstable.

Proof: Notice first that the multiplication of \mathbb{F} is an algebraic group and $(\mathbb{F}, \cdot, \Gamma)$ is of the Lang-type by A. 2 above. Thus by Proposition 2.6 of [6], we see that $T_{\mathbb{F}, \Gamma}$ is at least stable. $T_{\mathbb{F}, \Gamma}$ is in fact superstable since

1. the stability spectrum of $T_{\mathbb{F}, \Gamma}$ is the same as that of $T_{\Gamma(\mathbb{F})}$, the theory of restriction of $(\mathbb{F}, +, \cdot, \Gamma)$ to $\Gamma(\mathbb{F})$. Let $C \subset \mathbb{F}$. Observe first that there is only one complete 1-type over C in $T_{\mathbb{F}, \Gamma}$, which is realized by elements in $\mathbb{F} - \text{acl}_{\mathbb{F}}(\Gamma(\mathbb{F}) \cup C)$ where $\text{acl}_{\mathbb{F}}$ is the field-theoretic algebraic closure. Hence the cardinality of complete 1-types in $T_{\mathbb{F}, \Gamma}$ is bounded by the cardinality of the complete 1-types in $T_{\Gamma(\mathbb{F})}$. Thus they have the same stability spectrum.
2. $T_{\Gamma(\mathbb{F})}$ is superstable. For q transcendental, this is Theorem 1 in Section 5 of [15]. Combined with Proposition 10, it is easy to extend it to arbitrary q .

\blacksquare

3.4 Superstability of the line-bundle

In this subsection we show that the first-order theory of the line-bundle (\mathbf{U}, \mathbb{F}) is superstable.

Recall the $\mathcal{L}_{\omega_1, \omega}$ -sentence Ψ describing the properties of the quantum 2-torus $T_q^2(\mathbb{F})$. We now investigate the sentence Ψ from the first-order theoretic point of view.

Let $\mathcal{L}'_q = \mathcal{L}_q - \{\langle \cdot | \cdot \rangle\}$. Let $\text{Th}(\mathbf{U}, \mathbb{F})$ denote the first-order \mathcal{L}'_q -theory of the line bundle (\mathbf{U}, \mathbb{F}) . Unlike $\mathcal{L}_{\omega_1, \omega}$ -sentence Ψ , in $\text{Th}(\mathbf{U}, \mathbb{F})$ we can only say that Γ satisfies the first-order theory of a cyclic group with generator q .

Let M be a model of $\text{Th}(\mathbf{U}, \mathbb{F})$, then we have; for each $\gamma \in \Gamma$ and $(u, v) \in \mathbb{F}^* \times \mathbb{F}^*/\Gamma$, $\pi^{-1}(\gamma u, v)$, which is denoted as $\mathbf{U}_{(\gamma u, v)}$, is a subset of the Γ -bundle \mathbf{U} that is generated by an element. We also have $\mathbf{U}_{(\gamma u, v)} \subsetneq \mathbb{F}^* \mathbf{U}_{(\gamma u, v)}$ wher $\mathbb{F}^* \mathbf{U}_{(\gamma u, v)}$ is a subset of the line-bundle $\mathbb{F}^* \mathbf{U}$. Furthermore we have

$$\mathbf{U} = \bigcup_{\gamma, (u, v)} \mathbf{U}_{(\gamma u, v)}, \quad \mathbb{F}^* \mathbf{U} = \bigcup_{\gamma, (u, v)} \mathbb{F}^* \mathbf{U}_{(\gamma u, v)}.$$

Proposition 14 *The first-order theory of line-bundle (\mathbf{U}, \mathbb{F}) is superstable.*

Proof: We show that the first-order theory of line-bundle (\mathbf{U}, \mathbb{F}) is superstable in two steps;

1) show that the theory $\text{Th}(\mathbf{U}, \mathbb{F})$ of line-bundle (\mathbf{U}, \mathbb{F}) is *prime* over the theory $T_{\mathbb{F}, \Gamma}$; i.e., any isomorphism between two models of $T_{\mathbb{F}, \Gamma}$ can be extended to an isomorphism between two models of $\text{Th}(\mathbf{U}, \mathbb{F})$,

2) since the theory $T_{\mathbb{F}, \Gamma}$ is superstable (Theorem 13) the theory $\text{Th}(\mathbf{U}, \mathbb{F})$ of line-bundle (\mathbf{U}, \mathbb{F}) is superstable as well.

Proof of 1): Since the theory $T_{\mathbb{F}, \Gamma}$ is complete (Proposition 12), we may assume that $(\mathbb{F}_1, \Gamma) = (\mathbb{F}_2, \Gamma)$. We show that two saturated models M_1 and M_2 of $\text{Th}(\mathbf{U}, \mathbb{F})$ with the same cardinality are isomorphic as line-bundles over $\mathbb{F}^* \times \mathbb{F}^*/\Gamma$.

Let \mathbf{U}_1 and \mathbf{U}_2 denote the Γ -bundles over $\mathbb{F}^* \times \mathbb{F}^*/\Gamma$ in M_1 and M_2 respectively. Similarly, let $\mathbb{F}^* \mathbf{U}_1$ and $\mathbb{F}^* \mathbf{U}_2$ denote the line-bundles over $\mathbb{F}^* \times \mathbb{F}^*/\Gamma$ in M_1 and M_2 respectively.

Take $(u, v), (u', v') \in \mathbb{F}^* \times \mathbb{F}^*/\Gamma$. Consider vectors $\mathbf{u}(u, v) \in M_1$ and $\mathbf{u}(u', v') \in M_2$ respectively. The Γ -sets $\Gamma \cdot \mathbf{u}(u, v)$ in M_1 and $\Gamma \cdot \mathbf{u}(u', v')$ in M_2 are isomorphic. Therefore those 1-dimensional submodules generated by $\Gamma \cdot \mathbf{u}(u, v)$ and $\Gamma \cdot \mathbf{u}(u', v')$ are isomorphic as well. Hence we see that $\mathbb{F}^* \mathbf{U}_{(u, v)}$ in M_1 and $\mathbb{F}^* \mathbf{U}_{(u', v')}$ in M_2 are isomorphic.

Now by applying the operator V to $\mathbf{u}(u, v)$ in M_1 and $\mathbf{u}(u', v')$ in M_2 we move to other Γ -sets $\Gamma \cdot V(\mathbf{u}(u, v))$ in M_1 and $\Gamma \cdot V(\mathbf{u}(u', v'))$ in M_2 respectively. Then two 1-dimensional submodules generated by $\Gamma \cdot V(\mathbf{u}(u, v))$ and $\Gamma \cdot V(\mathbf{u}(u', v'))$ are isomorphic as well.

In this way we see that the Γ -bundle in M_1 and the Γ -bundle in M_2 are isomorphic. Then we extend this isomorphism to an isomorphism between the line-bundles M_1 and M_2 . This completes the proof of primeness.

Proof of 2): By the primeness shown in 1) we see that any realization of a type in $\text{Th}(\mathbf{U}, \mathbb{F})$ is fixed by automorphisms of models of $T_{\mathbb{F}, \Gamma}$, hence the cardinality of types in $\text{Th}(\mathbf{U}, \mathbb{F})$ is bounded by the cardinality of the types in $T_{\mathbb{F}, \Gamma}$. Therefore by Theorem 13, the theory $\text{Th}(\mathbf{U}, \mathbb{F})$ of line-bundle (\mathbf{U}, \mathbb{F}) is superstable. ■

It follows immediately that we have the following second main theorem;

- Theorem 15** 1. *The first-order theory of line-bundle (\mathbf{V}, \mathbb{F}) is superstable,*
 2. *The first-order theory of line-bundles $(\mathbf{U}, \mathbf{V}, \mathbb{F})$ is superstable.*

3.5 Arithmetic in the theory of quantum 2-torus

In this subsection we show that with the pairing function the ring of integers can be defined in Γ . In this regard it is similar to the theory of pseudo-exponentiation, the model theory of which can successfully be investigated “modulo arithmetic” as in [13] or [4].

First we may identify (Γ, \cdot) with $(\mathbb{Z}, +)$ via the correspondence

$$q^r \mapsto r.$$

This gives us immediately a definable addition $+$ on \mathbb{Z} by the exponential law.

A definable multiplication \times on Γ is defined as follows with the pairing function. Fix $u, v \in \mathbb{F}$, $\mathbf{u}(u, v) \in \mathbf{U}$, and $\mathbf{v}(v, u) \in \mathbf{V}$ which satisfy

$$\langle \mathbf{u}(u, v) | \mathbf{v}(v, u) \rangle = 1.$$

Then given $\alpha, \beta, \gamma \in \mathbb{Z}$, set by (9)

$$\alpha \times \beta = \gamma \quad \text{if and only if} \quad \langle q^\alpha \mathbf{u}(u, v) | q^\beta \mathbf{v}(v, u) \rangle = q^\gamma.$$

Let \oplus and \otimes be the pull-backs of $+$ and \times in $(\Gamma, \cdot, 1)$ respectively via the above correspondence. We then have

Proposition 16 *The two operations \oplus and \otimes defined above are commutative, i.e., for $\gamma_1, \gamma_2 \in \Gamma$ we have*

1. $\gamma_1 \oplus \gamma_2 = \gamma_2 \oplus \gamma_1$,
2. $\gamma_1 \otimes \gamma_2 = \gamma_2 \otimes \gamma_1$.

Theorem 17 (i) *With the pairing function, within $(\Gamma, \cdot, 1, q)$ we can define $(\Gamma, \oplus, \otimes, 1, q)$ and $(\Gamma, \oplus, \otimes, 1, q) \simeq (\mathbb{Z}, +, \cdot, 0, 1)$. Hence the theory of the quantum 2-torus $(\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F}, \Gamma)$ is undecidable and unstable.*

(ii) *The non-elementary theory of the quantum 2-torus $(\mathbf{U}, \mathbf{V}, \langle \cdot | \cdot \rangle, \mathbb{F}, \Gamma)$ over fixed $\Gamma = q^{\mathbb{Z}}$ is categorical in uncountable cardinalities.*

Proof: (i) This is essentially the Proposition above.

(ii) This is a direct corollary of the two statements

Claim 1. The non-elementary theory of (\mathbb{F}, Γ) , the field with a distinguished fixed subgroup, is categorical in uncountable cardinalities.

Claim 2. The quantum 2-torus $(\mathbf{U}, \mathbf{V}, \mathbb{F}, \Gamma)$ with the pairing function is prime over (\mathbb{F}, Γ) .

For Claim 1, note first that the first-order theory of \mathbb{F} is uncountable categorical since \mathbb{F} is algebraically closed. For $\Gamma \simeq q^{\mathbb{Z}}$, what we cannot state in the first-order theory is $x \in \Gamma \iff \exists n \in \mathbb{Z} (x = q^n)$. This is expressible in the non-elementary theory. Thus the non-elementary theory of (\mathbb{F}, Γ) is categorical in uncountable cardinalities.

Claim 2 is in fact part of the proof of Proposition 14. ■

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